From Incomplete Preferences to Ranking via Optimization

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Abstract: We consider methods for aggregating preferences that are based on the resolution of discrete optimization problems. The preferences are represented by arbitrary binary relations (possibly weighted) or incomplete paired comparison matrices. This incomplete case remains practically unexplored so far. We examine the properties of several known methods and propose one new method. In particular, we test whether these methods obey a new axiom referred to here as *Self-Consistent Monotonicity.* Some results are established that characterize solutions of the related optimization problems.

Keywords: Aggregation of preferences, ranking, paired comparisons, quadratic assignment problem, Kemeny median

1 Introduction

We consider methods for aggregating preferences that are based on the resolution of discrete optimization problems. For a review and references see Cook and Kress (1992), and Belkin and Levin (1990), and also David (1988) and Van Blokland-Vogelesang (1991). Some algorithmic aspects can be found in Barthélemy (1989) and Litvak (1982). The preferences are represented by arbitrary binary relations (possibly weighted) or incomplete paired comparison matrices. The outcome of an aggregation method is a set of "optimal" rankings (linear or weak orders) of the alternatives. Namely, a ranking is said to be optimal if it provides an extremum of some chosen objective function that expresses the connection (or proximity) between an arbitrary ranking and the original preferences. One special feature of the aggregation problem with incomplete preferences is that the Borda-like

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score is not any more a good rating index, since it does not take into account the number of comparisons and the strength of “opponents” of each alternative. The incomplete case remains practically unexplored so far. In this paper, we present some initial results concerning this problem.

We examine the properties of several known methods formulated for the incomplete case and propose one new method. In particular, we test whether these methods obey a new axiom, referred to as *Self-Consistent Monotonicity.* Our results suggest that the methods under consideration hardly satisfy this condition because of their discreteness. This paper provides only one example of methods satisfying Self-Consistent Monotonicity (Section 5). Actually, that method is an *indirect scoring procedure* rather than an aggregating operator based on discrete optimization. Indirect scoring procedures are considered in Chebotarev and Shamis (1996), where a sufficient condition of Self-Consistent Monotonicity and some more positive examples are given. A discussion of some other axioms for aggregating incomplete preferences can be found in Chebotarev and Shamis (1994).

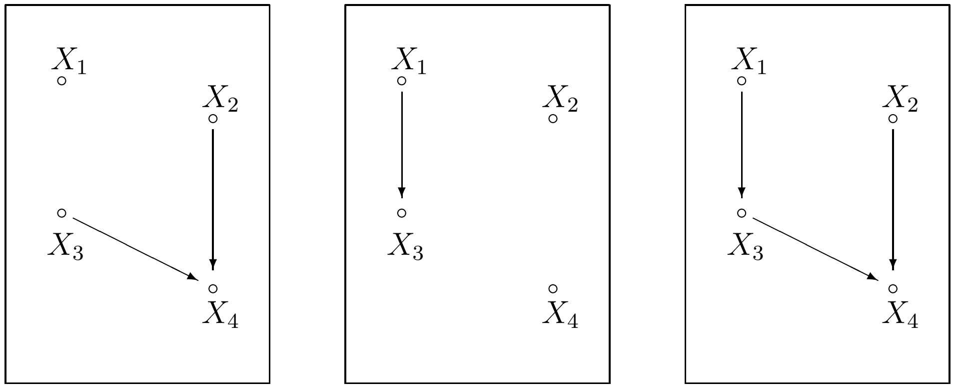
2 An illustrative example and the aims of the paper

We start with a simple example. Let there be four candidates and two voters. The preferences of these voters are incomplete. Namely, the first voter says: “X_2 and X_3 are better than X_4, and I know nothing about X_1.” (Fig. 1 a.) The second voter says: “X_1 is better than X_3, and I know nothing about X_2 and X_4.” (Fig. 1 b.)

Certainly, these preferences are extremely poor, and it is difficult to make a decision based on them. Nevertheless, having in mind more lifelike situations, we may pose a question: Which principles should be followed when aggregating incomplete preferences. Every answer will imply some consequences applicable to complete preferences as well.

Obviously, in this example X_1 or X_2 should be the first. We believe that X_1 has a small advantage over X_2, since X_1 defeats a “stronger” opponent. The rank order of the other candidates is more clear: $X_2 \succ X_3 \succ X_4$.

Our main requirement to the aggregating operators comes down to the following. Suppose we consider two alternatives, and the first alternative as compared to the second one
– achieves the better scores against the “stronger” opponents or
– achieves the better scores against the opponents of the same “strengths” or
– achieves the same scores against the “stronger” opponents.



a. The first voter b. The second voter c. combined (two voters)

FIGURE 1. Preferences of two voters

Then the first alternative should be placed *higher* than the second one in the social ranking.

Now we have to explain what is meant by "stronger". In the above requirement, "stronger" signifies "*is placed higher in the social ranking that is mentioned in the requirement.*" This requirement is formalized in Section 7 and is referred to as *Self-Consistency*. To obtain *Self-Consistent Monotonicity*, the main axiom in this paper, we require that if the first alternative additionally achieves some extra "wins" and/or the second alternative has extra "losses", then the first alternative should remain *higher* than the second one in the social ranking.

It turns out that very few methods satisfy these natural axioms. In this paper, we adjust a number of discrete optimization procedures to the case of incomplete preferences. The most familiar are the close procedures by Kemeny (1959) and Slater (1961) (their idea has been initially suggested by Kendall (1955)). For instance, the Slater method minimizes the number of arcs (which designate individual binary preferences) directed upwards (from a "worse" alternative to a "better" one) in the social ranking. When applied to the above example, this method produces three optimal social rankings (with no upward arcs), namely, $X_1 \succ X_2 \succ X_3 \succ X_4$, $X_2 \succ X_1 \succ X_3 \succ X_4$, and $X_1 \succ X_3 \succ X_2 \succ X_4$. Only the first of them preserves Self-Consistent Monotonicity.

The aims of this paper are:

– to collect some discrete optimization methods, to represent them in the assignment-like form (Section 4), and to define their modifications that generate weak orders (Section 9);

– to give some results characterizing solutions of these optimization problems (Section 6);
– to introduce Self-Consistent Monotonicity (Section 7);
– to present some necessary conditions of Self-Consistent Monotonicity (Sections 8 and 10);
– to prove Self-Consistent Monotonicity for the generalized row sum method and to test discrete optimization methods in this respect (Sections 8, 10, and 11);
– to outline the difficulties in obeying Self-Consistent Monotonicity by discrete optimization methods (Section 11).

3 Notation

The case of incomplete and possibly weighted preferences requires some more complex notation. Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of alternatives to be compared. To represent arbitrary preference relations of individuals, both ordinary binary relations and weighted ones, we use *incomplete paired comparison matrices.* Such a matrix of the pth individual ($p = 1, \ldots, m$) is an $n \times n$ table $R^{(p)} = (r_{ij}^p)_{i,j=1}^n$ whose entries r_{ij}^p and r_{ji}^p represent the result of comparing X_i to X_j by this individual. The ordered pair (r_{ij}^p, r_{ji}^p) will be called the *outcome* of that comparison. Here the value of r_{ij}^p can be interpreted as a measure of advantage of X_i over X_j (something like the number of scored goals in sport). If X_i and X_j have not been compared by that individual, the two corresponding cells of the table remain empty (i.e., r_{ij}^p and r_{ji}^p are undefined). The diagonal elements r_{ii}^p ($i = 1, \ldots, n$) do not correspond to any comparisons and are defined according to some convention. The collection of matrices $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ is called *an array of paired comparisons* of the alternatives $X_1, \ldots, X_n$. If all $R^{(p)}$ ($p = 1, \ldots, m$) are completely defined, then $\mathcal{R}$ is said to be *complete.*

Let $R = (r_{ij})$ be a complete $n \times n$-matrix, where

$$r_{ij} = \sum_{p|(i,j)} r_{ij}^p, \quad i, j = 1, \ldots, n$$

with summation over p for which r_{ij}^p is defined in $\mathcal{R}$. If for every p, r_{ij}^p is undefined in $R^{(p)}$, then, by definition, $r_{ij} = 0$. The total number of comparisons between X_i and $X_j \neq X_i$ will be denoted by m_{ij}:

$$m_{ij} = |\{p : r_{ij}^p \ \ is \ \ defined \ \ in \ \ \mathcal{R}\}|.$$

If $i = j$ then $m_{ij} = 0$.

An *aggregating operator* is a mapping assigning to every $\mathcal{R}$ with fixed $n \geq 2$ and $m \geq 1$ a nonempty set of weak orders on $\mathcal{X}$. These weak orders are called

optimal. If an aggregating operator always generates only *linear* optimal orders, we say that it is *strict.* Recall that a *weak order* is a complete and transitive binary relation; a *linear order* is an antisymmetric weak order. $\mathcal{W}$ and $\mathcal{L}$ will denote the set of all weak orders on $\mathcal{X}$ and the set of all linear orders on $\mathcal{X}$, respectively. These binary relations are considered here as those of preference, i.e., for such a relation ρ, $X_i\,\rho\,X_j$ means "X_i is not worse than X_j according to ρ."

For any binary relation ρ on $\mathcal{X}$ and for every $X_i \in \mathcal{X}$, the *Copeland index of X_i in ρ* can be defined as follows:

$$\rho(X_i) = \rho(i) = \rho\, i = |\{X_j \in \mathcal{X}:\ X_i\,\rho\,X_j\}| - |\{X_j \in \mathcal{X}:\ X_j\,\rho\,X_i\}|. \tag{1}$$

If ρ is a weak order, then $\rho\, i > \rho\, j$ can be interpreted as "X_i is better than X_j in ρ". In this case we write $X_i \succ_\rho X_j$. If ρ is a weak order and $\rho\, i = \rho\, j$, we say that X_i and X_j are *tied* in ρ and write $X_i \sim_\rho X_j$. The expression $X_i \succeq_\rho X_j$ denotes the disjunction of $X_i \succ_\rho X_j$ and $X_i \sim_\rho X_j$. Thus the Copeland index enables to extend the notion of rank to weak orders (in the manner like that used in statistics).

Paired comparisons can be dichotomous ($r_{ij}^p \in \{-1,1\}$ or $r_{ij}^p \in \{0,1\}$), with draws ($r_{ij}^p \in \{-1,0,1\}$ or $r_{ij}^p \in \{0,\frac{1}{2},1\}$), numerical, and so on; different connections between r_{ij}^p and r_{ji}^p can be imposed. In this paper, we consider incomplete paired comparisons with the only connection between r_{ij}^p and r_{ji}^p that if r_{ij}^p is defined in $\mathcal{R}$, then r_{ji}^p is defined too. Let us suppose that there exist $r_{\min}$ and $r_{\max} > r_{\min}$ such that all entries r_{ij}^p must belong to the closed interval $[r_{\min}, r_{\max}]$. An outcome (r_{ij}^p, r_{ji}^p) of comparing X_i to X_j will be called a *maximal win* if $r_{ij}^p = r_{\max}$ and $r_{ji}^p = r_{\min}$; it will be called a *maximal loss* if $r_{ij}^p = r_{\min}$ and $r_{ji}^p = r_{\max}$. As it has been mentioned above, we do not require that $\mathcal{R}$ consists of only maximal wins and maximal losses, however our results are applicable to the paired comparisons of that type as well. We only suppose that maximal wins (maximal losses) *are admissible.* By definition, put $r_{ii}^p = 0$ for all $i \in \{1,\dots,n\}$ and $p \in \{1,\dots,m\}$.

Now we introduce the *Copeland index $t(X_i)$ of X_i in the array of paired comparisons $\mathcal{R}$*:

$$t(X_i) = t_i = \sum_{(j,p)|i} (r_{ij}^p - r_{ji}^p) = \sum_{j=1}^{n} (r_{ij} - r_{ji}), \tag{2}$$

where $(j,p)|i$ denotes summation over j and p, for which r_{ij}^p is defined in $\mathcal{R}$.

Remark. This framework tolerates many diverse ways of extracting the numbers r_{ij}^p from the individual perceptions. In this paper, we confine ourselves to the data for which sums and differences (such as in the above formula) make sense. This means that they are compatible with the scale type or are meaningful in some other exact model. Below we examine the properties of different objective functions based on these operations.

4 Objective functions for aggregating preferences

It can be easily shown that many known optimization methods for aggregating preferences can be reduced to *quadratic assignment problems* of the form

$$\mathrm{QA}(R,C): \quad \textit{maximize} \ \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} C(\rho\, i, \rho\, j) \ \textit{ for } \ \rho \in \mathcal{P}, \tag{3}$$

where $C(\cdot,\cdot)$ is a fixed *structure function*, and $\rho\, i$ is the Copeland index of X_i in ρ. Now $\mathcal{P}$ is the set of all *linear orders* on $\mathcal{X}$ (i.e., $\mathcal{P} = \mathcal{L}$), but below a more general case ($\mathcal{P} = \mathcal{W}$) is considered too. The quadratic assignment objective function measures some multiplicative consistency (depending on $C(\cdot,\cdot)$) between the original preferences and a tentative resulting order ρ.

The formulation (3) of the quadratic assignment problem is not conventional. We use the Copeland index of X_i in ρ instead of a simple rank (see, e.g., Hubert (1976) and Arditti (1983)) since this straightforward generalization provides an easy way to introduce *weak quadratic assignment problems* (WQA(R,C)) involving arbitrary *weak orders* ($\mathcal{P} = \mathcal{W}$) instead of *linear orders* on $\mathcal{X}$.

As long as quadratic assignment problems with $\mathcal{P} = \mathcal{L}$, are considered, the following structure functions are relevant:

$$C_1(x,y) = \mathrm{sign}(x-y), \quad C_2(x,y) = (\mathrm{sign}(x-y))^{(+)}, \quad C_3(x,y) = (\mathrm{sign}(x-y))^{(-)},$$

$$C_4(x,y) = x-y, \quad C_5(x,y) = (x-y)^{(+)}, \quad C_6(x,y) = (x-y)^{(-)},$$

where $z^{(+)} = \max(z,0)$, $z^{(-)} = \min(z,0)$, and

$$\mathrm{sign}\, z = \begin{cases} -1, & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ 1, & \text{if } z > 0. \end{cases}$$

Note that $C_1(x,y) = C_2(x,y)+C_3(x,y)$, $C_4(x,y) = C_5(x,y)+C_6(x,y)$, $C_2(x,y) = \mathrm{sign}(C_5(x,y))$, $C_3(x,y) = \mathrm{sign}(C_6(x,y))$.

In the following list of objective functions (and of corresponding methods) structure functions $C_1,\dots,C_6$ are used not only for quadratic assignment problems.

1. Three distinct extensions of the Slater (1961) method, which had been originally suggested by Kendall (1955): QA(R,C_k), $k \in \{1,2,3\}$ (see, e.g., Arditti (1983)).

2. Three distinct extensions of the Kemeny (1959) method (which is equivalent to the Slater method in the complete dichotomous case):

$$minimize \sum_{(i,j,p)|\mathcal{R}} |r_{ij}^p - C_k(\rho\, i, \rho\, j)| \;\; for \;\; \rho \in \mathcal{P}$$

with $k \in \{1, 2, 3\}$ and summation over those i, j, p for which r_{ij}^p is defined in $\mathcal{R}$. According to Young (1986), Kemeny's method had been initially proposed in a vague form by Condorcet.

3. "Weighted sum of back scores": QA(R, C_6). This method was suggested by Thompson (1975) and Hubert (1976), and studied in Kano and Sakamoto (1985), and Frey and Yehia-Alcoutlabi (1986).

4. "Weighted sum of right scores": QA(R, C_5) (Kano and Sakamoto (1983)).

5. "Weighted sum of all scores": QA(R, C_4) (Chebotarev (1988, 1990), Crow (1990)). This method can be reduced to ordering alternatives by "sum of wins minus sum of losses" (it is the Copeland index; see Theorem 1 below) and is connected to some ideas of Kendall (1970).

6. "Net sum of back scores" (see, e.g., Weiss and Assous (1987), Crow (1990)) QA$((R - R^T)^{(+)}, C_3)$:

$$maximize \sum_{i=1}^{n} \sum_{j:\, m_{ij}>0} (r_{ij} - r_{ji})^{(+)} C_3(\rho\, i, \rho\, j) \; for \; \rho \in \mathcal{P}\,.$$

7. The following four methods are based on the idea of balancing "back scores" of two types: "wins above" and "losses below" (Crow (1990, 1993)).

7a. Sum of absolute differences between Wins Above and Losses Below – "WALB":

$$minimize \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} (r_{ij} C_3(\rho\, i, \rho\, j) - r_{ji} C_3(\rho\, j, \rho\, i)) \Big| \; for \; \rho \in \mathcal{P}\,.$$

7b. "Refined WALB":

$$minimize \sum_{i=1}^{n} \Big| \sum_{j:\, m_{ij}>0} \frac{1}{m_{ij}} (r_{ij} C_3(\rho\, i, \rho\, j) - r_{ji} C_3(\rho\, j, \rho\, i)) \Big| \; for \; \rho \in \mathcal{P}\,.$$

7c. "Net WALB":

$$minimize \sum_{i=1}^{n} \Big| \sum_{j:\, m_{ij}>0} ((r_{ij} - r_{ji})^{(+)} C_3(\rho\, i, \rho\, j) - (r_{ji} - r_{ij})^{(+)} C_3(\rho\, j, \rho\, i)) \Big| \; for \; \rho \in \mathcal{P}\,.$$

7d. "Refined Net WALB":

$$minimize \sum_{i=1}^{n} \Big| \sum_{j:\, m_{ij}>0} \frac{1}{m_{ij}} ((r_{ij} - r_{ji})^{(+)} C_3(\rho\, i, \rho\, j) - (r_{ji} - r_{ij})^{(+)} C_3(\rho\, j, \rho\, i)) \Big| \; for \; \rho \in \mathcal{P}\,.$$

7e. "Net-Difference-WALB":

$$minimize \sum_{i=1}^{n} \Big| \sum_{j:\, m_{ij}>0} ((r_{ij} - r_{ji})^{(+)} C_6(\rho\, i, \rho\, j) - (r_{ji} - r_{ij})^{(+)} C_6(\rho\, j, \rho\, i)) \Big| \; for \; \rho \in \mathcal{P}\,.$$

The following method is new.

8. "β-Least-Squares" (β-LS): $minimize \sum_{(i,j,p)|\mathcal{R}} (r_{ij}^p - \beta C_4(\rho\, i, \rho\, j))^2 \; for \; \rho \in \mathcal{P}$

with summation over those i, j, p for which r_{ij}^p is defined in $\mathcal{R}$. Here β is a positive real parameter.

Remark. In the methods based on "net scores" (i.e., "Net sum of back scores", "Net WALB", "Refined Net WALB", and "Net-Difference-WALB") a "net draw" ($m_{ij} > 0,\ r_{ij} = r_{ji}$) between two alternatives with different positions in ρ ($\rho\, i \neq \rho\, j$) is worth being distinguished from the lack of comparisons between them ($m_{ij} = 0$, where $r_{ij} = r_{ji}$ by definition). If only maximal wins/losses are allowed, then the following modification provides this distinction: replace $(r_{ij} - r_{ji})^{(+)}$ by $\psi(r_{ij} - r_{ji})$, where

$$\psi(z) = \begin{cases} z, & \text{if } z > 0, \\ (r_{\max} - r_{\min})/2, & \text{if } z = 0, \\ 0, & \text{if } z < 0, \end{cases}$$

as Crow (1990, 1993) proposes for the case $r_{\max} = 1,\ r_{\min} = 0$. Note that such a modification preserves our results that involve these methods, i.e., Corollary 1 and Theorem 6 below. Another possible modification based on the function $\psi'(z) = (z + r_{\max} - r_{\min})^{(+)}$ preserves Corollary 1 and Theorem 6 as well. Above we wrote $j : m_{ij} > 0$ under the second sums in the objective functions of the "Net"-methods in order to support these possible modifications.

Some other methods can be obtained by extending the measures of association from Critchlow (1985) to incomplete paired comparisons.

5 Generalized row sum method

The generalized row sum method (Chebotarev (1989, 1994)) is not based on the resolution of a discrete optimization problem, however it has some connection with the β-LS method (Theorem 3 below). On the other hand, the generalized row sum method will be shown to satisfy Self-Consistent Monotonicity, the main axiom in this paper (Theorem 8).

For the sake of simplicity, we suppose here that incomplete paired comparison matrices $R^{(p)}$, $p = 1, \ldots, m$, are skew-symmetric: if r^p_{ij} is defined in $R^{(p)}$, then $r^p_{ji} = -r^p_{ij}$. In this case, $r_{\min} = -r_{\max}$. The generalized row sum method estimates the alternatives by the indexes $x_1, \ldots, x_n$ (*generalized row sums*) that satisfy the following system of linear equations:

$$x_i = \sum_{(k,p)|i} (r^p_{ik} + \varepsilon \cdot (x_k - x_i + r^p_{ik} mn)), \quad i = 1, \ldots, n, \tag{4}$$

where ε is a nonnegative parameter. This system of equations has been proven to have a unique solution for every $\mathcal{R}$. The corresponding optimal weak order ρ is defined as follows: $X_i \succ_\rho X_j$ iff $x_i > x_j$.

The generalized row sum method is an extension of the row sum method (and of the Borda rule in the case where individual preferences are linear orders) to

incomplete paired comparisons. Specifically, if $\mathcal{R}$ is complete, then for any $\varepsilon \geq 0$, $x_i = s_i$ $(i = 1, \ldots, n)$ holds, where

$$s_i = \sum_{(k,p)|i} r^p_{ik} = t_i/2.$$

This method has been derived both axiomatically and statistically. The value $f^p_{ik} = r^p_{ik} + \varepsilon\cdot(x_k - x_i + r^p_{ik}mn)$ is the contribution of the comparison outcome r^p_{ik} to the estimate x_i of X_i. Parameter $\varepsilon \geq 0$ is said to be *reasonable* for given n and m if for any array $\mathcal{R}$ that consists of m n-by-n paired comparison matrices, the value

$$f^p_{ik} = r^p_{ik} + \varepsilon\cdot(x_k - x_i + r^p_{ik}mn)$$

is non-negative at $r^p_{ik} = r_{\max}$ (maximal win) and non-positive at $r^p_{ik} = r_{\min} = -r_{\max}$ (maximal loss), for any i, j, and p.

It has been shown that the reasonableness of ε is equivalent to satisfying the constraint

$$0 \leq \varepsilon \leq \frac{1}{m(n-2)}.$$

6 Some connections to direct methods

In this section, we prove three theorems concerning connections between discrete optimization methods, namely QA(R, C_4) and β-LS, and direct methods for aggregating preferences. The first two theorems are formulated for the general case of weak orders ($\mathcal{P} = \mathcal{W}$). Theorem 1 shows that the method QA(R, C_4) can be reduced to ordering alternatives in the decreasing order of their Copeland indexes (with an arbitrary order of the alternatives having the same Copeland index). Note that the related problems QA(R, C_5) and QA(R, C_6) are, in general, NP-complete.

Theorem 1 (Reduction of WQA(R, C_4) to the Copeland ranking): *A weak order ρ is a solution of* WQA(R, C_4) *for $\mathcal{R}$ if and only if*

$$\textit{for any } X_i, X_j \in \mathcal{X}, \quad t_i > t_j \textit{ implies } X_i \succ_\rho X_j. \tag{5}$$

The proofs of all statements are given in the Appendix. An analogous theorem for *linear* orders has been proved in Chebotarev (1988, 1990).

A similar statement holds for the β-LS method with small enough β.

Theorem 2 (Partial reduction of β-LS with small β to the Copeland ranking): *Let $\mathcal{R}$ be an array of paired comparisons on $\mathcal{X}$. There exists a number $\beta_0 > 0$*

such that: if $0 < \beta < \beta_0$*, then every solution* $\rho^* \in \mathcal{W}$ *of the* β*-LS problem with parameter* β *for* $\mathcal{R}$ *satisfies the following condition:*

$$\textit{for any } X_i, X_j \in \mathcal{X}, \quad t_i > t_j \textit{ implies } X_i \succ_{\rho^*} X_j. \tag{6}$$

There is an important difference between the methods QA(R, C_4) and β-LS with a small β. Namely, according to Theorem 2, (6) is a necessary but not a sufficient condition of optimality. In other words, the latter method does not permit arbitrariness in ordering the alternatives with the same Copeland index. Indeed, it can be easily shown by examples that β-LS with a small parameter may yield a narrower set of optimal orders than QA(R, C_4).

Now consider a continuous counterpart of the β-Least-Squares method. Note that for any linear order ρ on $\mathcal{X}$, $\sum_{i=1}^{n} \rho\, i = 0$ and $\sum_{i=1}^{n} (\rho\, i)^2 = \frac{1}{3}(n-1)n(n+1)$. Denote the latter value by D_n^2 and consider the following *relaxed* β*-LS method:*

$$\textit{minimize} \quad \sum_{(i,j,p)|\,\mathcal{R}} (r_{ij}^p - \beta(y_i - y_j))^2 \ \textit{ for real } \ y_1, \ldots, y_n \tag{7}$$

subject to

$$\sum_{i=1}^{n} y_i = 0 \tag{8}$$

and

$$\sum_{i=1}^{n} y_i^2 = D_n^2. \tag{9}$$

The difference between β-LS and relaxed β-LS is that for the former problem the set of admissible solutions is narrower: not the whole intersection of the hyperplane $\sum_{i=1}^{n} y_i = 0$ with the hypersphere $\sum_{i=1}^{n} y_i^2 = D_n^2$, but the set of points obtained from $(-(n-1), -(n-3), \ldots, (n-1))$ by all possible permutations of the coordinates. These points are all vertices of a specific polyhedron (polytope) inscribed into that intersection. According to the following theorem, relaxed β-LS is closely connected to the generalized row sum method.

THEOREM 3 (Reduction of the relaxed β-LS to the generalized row sums): *Let* $y = (y_1, \ldots, y_n)$ *be a solution of the relaxed* β*-LS problem with some* β *for an array of paired comparisons* $\mathcal{R} = (r_{ij}^p)_{i,j\in\{1,\ldots,n\}}^{p\in\{1,\ldots,m\}}$*. Let* $\mathcal{R}'$ *be an array of paired comparisons with elements* $(r_{ij}^p)' = r_{ij}^p - r_{ji}^p$*. Then for some* ε*, the vector* y *is proportional to the vector of generalized row sums obtained with parameter* ε *for* $\mathcal{R}'$*.*

Another example of Lagrangian relaxation applied to a discrete preference aggregation problem can be found in Arditti (1983).

7 Self-Consistency and Self-Consistent Monotonicity

H.A. David (1987) said "...nonparametric method cannot be entirely satisfactory when the m_{ij} differ greatly." Our aim is to investigate to what extent such a method can be satisfactory, and so we examine the properties of the methods above. In this section, a new axiom named *Self-Consistency* and its extension, *Self-Consistent Monotonicity* are introduced.

Let us say that an outcome (r^p_{ik}, r^p_{ki}) of comparing X_i to X_k is *not weaker with respect to a weak order* ρ than an outcome $(r^q_{j\ell}, r^q_{\ell j})$ of comparing X_j to X_ℓ iff $r^p_{ik} \geq r^q_{j\ell}$, $r^p_{ki} \leq r^q_{\ell j}$, and $X_k \succeq_\rho X_\ell$. If, in addition, at least one of the inequalities (relations) is strict, then the outcome (r^p_{ik}, r^p_{ki}) is said to be *stronger* than $(r^q_{j\ell}, r^q_{\ell j})$ with respect to ρ.

Self-Consistency. For any optimal weak order ρ and for any $X_i, X_j \in \mathcal{X}$, the statement [There exists a one-to-one correspondence between the set of comparison outcomes of X_i and the set of comparison outcomes of X_j such that each outcome of X_i is not weaker than the corresponding outcome of X_j with respect to ρ] implies $[X_i \succeq_\rho X_j]$. If, in addition, at least one outcome of X_i is stronger than the corresponding outcome of X_j with respect to ρ, then $X_i \succ_\rho X_j$.

Self-Consistency enables us to confront two alternatives having the same number of comparisons. Now suppose that the alternative dominating in such a confrontation achieves several extra *maximal wins* and the dominated alternative gets some number of extra *maximal losses*. It is reasonable to demand that this addition of extra outcomes preserves the result of confrontation: the former alternative remains "better". Let us extend Self-Consistency in this way.

Self-Consistent Monotonicity (SCM). Suppose ρ is an optimal weak order and $X_i, X_j \in \mathcal{X}$. Let $\mathcal{R}_i$ and $\mathcal{R}_j$ be the sets of comparison outcomes of X_i and X_j, respectively. Suppose that $\mathcal{R}_i = \mathcal{R}'_i \cup \mathcal{R}''_i$ ($\mathcal{R}_i \cap \mathcal{R}''_i = \emptyset$), $\mathcal{R}_j = \mathcal{R}'_j \cup \mathcal{R}''_j$ ($\mathcal{R}_j \cap \mathcal{R}''_j = \emptyset$), $\mathcal{R}''_i$ consists of maximal wins, $\mathcal{R}''_j$ consists of maximal losses, and there exists a one-to-one correspondence between $\mathcal{R}'_i$ and $\mathcal{R}'_j$ (in particular, $\mathcal{R}'_i$ and $\mathcal{R}'_j$ may be empty) such that every outcome from $\mathcal{R}'_i$ is not weaker than the corresponding outcome from $\mathcal{R}'_j$ with respect to ρ. Then $X_i \succeq_\rho X_j$. If, in addition, at least one outcome from $\mathcal{R}'_i$ is stronger than the corresponding outcome from $\mathcal{R}'_j$ with respect to ρ or $\mathcal{R}''_i \neq \emptyset$ or $\mathcal{R}''_j \neq \emptyset$, then $X_i \succ_\rho X_j$.

Possibly, some analysts can be inclined to consider the entire set of optimal orders as an indivisible macro-decision whose elements represent different characteristic features of the set of original preferences. From this point of view, optimal orders should be considered not separately but jointly, and Self-Consistency which

addresses to every separate optimal order is a surplus requirement. A possible objection to this opinion is as follows. In most situations we have to make only one decision. As soon as it is made, any appealing to other optimal decisions becomes out of place. The decision we make should be logical by itself, apart from rejected opportunities.

8 All strict operators break Self-Consistency

Recall that an aggregating operator is *strict* if its optimal orders are always linear.

THEOREM 4: *If an aggregating operator is strict, then it does not satisfy Self-Consistency.*

This theorem has an easy but somewhat degenerate proof. Indeed, note that Self-Consistency does not prohibit the sets of comparison outcomes of X_i and X_j to be empty. In this case, Self-Consistency implies $X_i \succeq_\rho X_j$ and $X_j \succeq_\rho X_i$, which is broken by any linear order. A similar proof with alternatives that have nonempty sets of comparisons and $n > 2$ can be carried out by considering the following $\mathcal{R}$: $r_{13}^1 = r_{23}^1 = r_{\max}$, $r_{31}^1 = r_{32}^1 = r_{\min}$; all other r_{ij}^p with $i \neq j$ are undefined. In the Appendix we give another proof, which demonstrates the application of Self-Consistency to cyclic preferences.

9 Operators generating weak orders

Theorem 4 motivates the consideration of aggregating operators that generate not only linear orders but arbitrary weak orders. In particular, we shall consider *weak quadratic assignment problems* WQA(R, C), i.e., problems (3) with $\mathcal{P} = \mathcal{W}$ (note that Theorem 1 and Theorem 2 have been formulated for this general case).

To that end it is useful to modify structure functions C_2, C_3, C_4, and C_6. Indeed, note that the structure functions $C_1(x, y), \ldots, C_6(x, y)$ depend on $x - y$. Suppose $g(d)$ is the contribution of the comparison outcome ($r_{ij}^p = r_{\max}$, $r_{ji}^p = r_{\min}$) to the quadratic assignment objective function, provided that $\rho\, i - \rho\, j = d$. Then, by (3), $g(d) = r_{\max} C(d, 0) + r_{\min} C(0, d)$. It is reasonable to require

$$g(-1) < g(0) < g(1). \qquad (10)$$

Indeed, since the quadratic assignment objective function measures consistency between the original preferences and a tentative resulting order, this requirement

is motivated by that maximal win is more natural for alternatives with higher social estimate.

For C_1 and C_4, (10) amounts to $r_{\max} > r_{\min}$, whereas for C_2, C_3, C_5, and C_6 it is equivalent to $[r_{\max} > 0$ and $r_{\min} < 0]$. Therefore (10) is broken even for the customary sporting point systems: $r_{ij}^p \in \{0, \frac{1}{2}, 1\}$ and $r_{ij}^p \in \{0, 1, 2\}$. As a result, for these point systems, the weak order in which all alternatives are tied is never optimal for $\mathrm{QA}(R, C_2)$ and $\mathrm{QA}(R, C_5)$, and is always optimal for $\mathrm{QA}(R, C_3)$ and $\mathrm{QA}(R, C_6)$.

Thus let us revise C_2, C_3, C_5, and C_6 as follows:

$$C_2'(x, y) = \mathrm{sign}(x - y) + 1, \quad C_3'(x, y) = \mathrm{sign}(x - y) - 1,$$

$$C_5'(x, y) = (x - y + 1)^{(+)}, \quad C_6'(x, y) = (x - y - 1)^{(-)}.$$

C_1 and C_4 do not require revisions: let $C_1'(x, y) = C_1(x, y), \quad C_4'(x, y) = C_4(x, y)$.

For all these functions, (10) amounts to $r_{\max} > r_{\min}$, and they are equivalent to their prototypes in all optimization methods of Section 4 in the strict case. In the rest of the paper, we consider quadratic assignment problems and other problems of Section 4 with C_k' substituted for C_k $(k \in \{1, \ldots, 6\})$ and $\mathcal{P} = \mathcal{W}$.

10 Indifference to the degree of resulting preferences contradicts SCM

Let us say that an aggregating operator *equalizes* weak orders ρ and ρ' for $\mathcal{R}$ if ρ and ρ' are both optimal for $\mathcal{R}$ or both are not optimal. An aggregating operator will be called *indifferent to the degree of resulting preferences* if it equalizes every ρ and ρ' such that

$$\textit{for all } \; r_{ij}^p \; \textit{defined in } \; \mathcal{R}, \; \; \mathrm{sign}(\rho\, i - \rho\, j) = \mathrm{sign}(\rho'\, i - \rho'\, j).$$

Theorem 5: *If a nonstrict aggregating operator is indifferent to the degree of resulting preferences and $n > 2$, then it violates SCM.*

Corollary 1: *The nonstrict aggregating operators corresponding to:* $\mathrm{WQA}(R, C_k'),\ k \in \{1, 2, 3\}$; minimize $\sum\limits_{(i,j,p)\,|\,\mathcal{R}} |r_{ij}^p - C_k'(\rho\, i, \rho\, j)|$ for $\rho \in \mathcal{W},\ k \in \{1, 2, 3\}$ *(extensions of the Kemeny median); "Net sum of back scores", "WALB", "Net WALB", "Refined WALB", and "Refined Net WALB" violate SCM.*

11 Are there discrete optimization methods that obey Self-Consistent Monotonicity?

Theorem 6: *If $n > 2$, then the nonstrict aggregating operators corresponding to* $\mathrm{WQA}(R, C'_k)$ *with* $k \in \{4, 5, 6\}$ *and "Net-Difference-WALB" violate SCM.*

The claim that the β-LS operator satisfies SCM might provide a "happy end" of this paper. However, this is not the case.

Theorem 7: *If $n > 4$ then the β-LS operator violates Self-Consistency for any $\beta > 0$.*

Recall that the β-LS method can be considered as a discrete analog of the generalized row sum method (Theorem 3).

Theorem 8: *The generalized row sum method with positive ε satisfies Self-Consistency. Moreover, it satisfies Self-Consistent Monotonicity when ε is positive and reasonable.*

Comparison of Theorem 7 and Theorem 8 suggests that the β-LS method fails to satisfy Self-Consistency because of its discreteness. Indeed, in the proof of Theorem 7 given in the Appendix, X_1 has a small superiority over X_2 in the original preferences, and Self-Consistency requires $X_1 \succ_\rho X_2$. However, β-LS ties X_1 and X_2 for every $n > 5$. Note that β-LS minimizes some kind of proximity between the initial preferences and the tested weak orders. The superiority of X_1 over X_2 turns out to be so small that it is closer to "draw" than to "win". This is typical of nonstrict discrete methods like β-LS. There are only three possible relations between two alternatives in a social weak order, "worse", "better", and "equivalent", and the latter turns out to be optimal for small superiorities under the nonstrict aggregating procedures. Is this a shortcoming or not? We believe that in case we must choose only one alternative, even a small superiority is worth being taken into account, and so such a tie is not useful. An advantage of continuous approaches is that they enable one to measure the differences between the adjacent alternatives, whereas the discrete methods give no means for that. (However, some information can be extracted through comparing the optimal value of the objective function with its values for orders where these alternatives are tied or interchanged.)

All the discrete optimization methods we considered proved to break Self-Consistent Monotonicity. Nevertheless, the question in the heading of this section is a methodological rather than a mathematical one. Indeed, a discrete optimization method that satisfies SCM can be designed artificially, for example, by using

explicit expressions of the generalized row sums $x_1, \ldots, x_n$:

$$\textit{maximize} \ \ \sum_{i=1}^{n}(x_i\,\rho\,i - \alpha(\rho\,i)^2) \quad \textit{for} \ \ \rho \in \mathcal{W}, \tag{11}$$

where α is a small enough positive constant.

To prove that the optimal values $\rho\,i$ are ordered exactly as x_i, note that every maximizing weak order for the objective function $\sum\limits_{i=1}^{n} x_i\,\rho\,i$ preserves the strict component of the order of $x_1, ..., x_n$ (Lemma 1 in the proof of Theorem 1), and the subtraction of $\alpha(\rho\,i)^2$ in (11) provides equal $\rho\,i$ for the alternatives with equal generalized row sums x_i. (Indeed, equal numbers provide a minimum for the sum of squares subject to their fixed sum.) It follows that aggregating operator (11) satisfies SCM. However, such a method would remain essentially based on "continuous" indexes. Now we do not know any proper discrete optimization operators that satisfy SCM.

12 Conclusion

If an aggregating operator is strict, then it breaks Self-Consistent Monotonicity (SCM), since this axiom prescribes equivalence of some alternatives (Theorem 4). Many aggregating operators associated with discrete optimization problems are "indifferent to the degree of resulting preferences", which is incompatible with SCM (Theorem 5). Nonstrict discrete optimization methods like β-LS violate Self-Consistent Monotonicity, since they produce equivalence of some alternatives, one of which having a small superiority over another. On the other hand, there are "continuous" methods that satisfy SCM, for example, the generalized row sum method (Theorem 8).

The transfer from linear orders to weak orders is the first step of relaxation. Possibly, this step is not sufficient for such a keen type of data as unbalanced (incomplete) preferences. A next possible step is the conversion to aggregation models with real unknown parameters that measure the value (utility) of alternatives. Such *indirect scoring procedures* are considered in Chebotarev and Shamis (1996) where a sufficient condition of Self-Consistent Monotonicity and some more positive examples are given.

Appendix: Proofs

PROOF OF THEOREM 1: Suppose ρ is an arbitrary weak order on $\mathcal{X}$, and $f(\rho)$ is

the value of the objective function for ρ. Then

$$f(\rho) = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}(\rho\, i - \rho\, j) = \sum_{i=1}^{n}\left(\rho\, i \sum_{j=1}^{n} r_{ij}\right) - \sum_{j=1}^{n}\left(\rho\, j \sum_{i=1}^{n} r_{ij}\right) = \sum_{i=1}^{n} \rho\, i\, t_i. \quad (A1)$$

Now it suffices to prove the following lemma.

Lemma 1 (A weak order maximizes scalar product iff it preserves relation ">"): *For any real vector $u = (u_1, \ldots, u_n)$, a weak order ρ is a solution of the problem*

$$\textit{maximize} \quad \sum_{i=1}^{n} u_i\, \rho\, i \quad \textit{for} \ \ \rho \in \mathcal{W} \qquad (A2)$$

if and only if

$$\textit{for any } X_i, X_j \in \mathcal{X}, \quad u_i > u_j \textit{ implies } X_i \succ_\rho X_j. \qquad (A3)$$

Proof of Lemma 1: Let ρ be a solution of the problem (A2). Assume that there exist X_k and X_ℓ such that $u_i > u_j$, but $\rho\, k \le \rho\, \ell$. Consider two cases.

(A) $\rho\, k < \rho\, \ell$. Consider the weak order ρ' that is obtained from ρ by interchanging X_k and X_ℓ. Then, according to (A1),

$$f(\rho) - f(\rho') = \sum_{i=1}^{n}(\rho\, i - \rho'\, i)u_i = (\rho\, k)u_k + (\rho\, \ell)u_\ell - (\rho'\, k)u_k - (\rho'\, \ell)u_\ell$$

$$= (\rho\, k)u_k + (\rho\, \ell)u_\ell - (\rho\, \ell)u_k - (\rho\, k)u_\ell = (\rho\, k - \rho\, \ell)(u_k - u_\ell) < 0,$$

and ρ cannot be a solution of QA(R, C_4), in contradiction to the assumption.

(B) $\rho\, k = \rho\, \ell$. Let $\mathcal{X}_{k\ell} = \{X_j : \rho\, j = \rho\, k\} \setminus \{X_k, X_\ell\}$. Recall that ρ is a binary relation, i.e., $X_v \succeq_\rho X_w$ is a designation of $(X_v, X_w) \in \rho$. We shall "move apart" X_k and X_ℓ in ρ preserving the positions of all other alternatives. Consider the weak order ρ' that is obtained by removing from ρ the pair (X_ℓ, X_k) and the pairs (X_j, X_k) and (X_ℓ, X_j) for all $X_j \in \mathcal{X}_{k\ell}$. Then $\rho'\, k = \rho\, k + |\, \mathcal{X}_{k\ell}\,| + 1$, $\rho'\, \ell = \rho\, \ell - |\, \mathcal{X}_{k\ell}\,| - 1$, and $\rho\, i' = \rho\, i$ for all $X_i \in \mathcal{X} \setminus \{X_k, X_\ell\}$. Hence, by (A1),

$$f(\rho) - f(\rho') = \sum_{i=1}^{n}(\rho\, i - \rho'\, i)u_i = (-|\, \mathcal{X}_{k\ell}\,| - 1)u_k + (|\, \mathcal{X}_{k\ell}\,| + 1)u_\ell$$

$$= (u_\ell - u_k)(|\, \mathcal{X}_{k\ell}\,| + 1) < 0,$$

and ρ cannot be a solution of QA(R, C_4), in contradiction to our assumption. Necessity of (A3) is shown.

To prove sufficiency of (A3), note that all weak orders satisfying (A3) can be obtained one from another by sequential adding and removing pairs (X_k, X_ℓ) such

that $u_k = u_\ell$. Therefore, if for an arbitrary i we denote the set $\{X_j \in \mathcal{X} : u_j = u_i\}$ by $\mathcal{X}_i$, then the value $\sum\limits_{X_j \in \mathcal{X}_i} \rho\, j$ is the same for all weak orders satisfying (A3). Consequently, all such weak orders have the same (and thus maximal!) value of (A1). This completes the proof of Lemma 1 and Theorem 1.

Proof of Theorem 2: Suppose ρ^* satisfies (6) and let for any ρ

$$f(\rho) = \sum_{(i,j,p) | \mathcal{R}} (r_{ij}^p - \beta C_4(\rho\, i, \rho\, j))^2$$

$$= \sum_{(i,j,p) | \mathcal{R}} (r_{ij}^p)^2 - 2\beta \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} C_4(\rho\, i, \rho\, j) + \beta^2 \sum_{(i,j,p) | \mathcal{R}} (\rho\, i - \rho\, j)^2. \qquad (A4)$$

The second term in the right-hand side of (A4) corresponds to $\mathrm{QA}(R, C_4)$. By Theorem 1, there exists a number $E > 0$ such that for any ρ' not satisfying (6),

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} C_4(\rho^*\, i, \rho^*\, j) - \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} C_4(\rho'\, i, \rho'\, j) > E.$$

Let F be an upper bound of $\sum\limits_{(i,j,p) | \mathcal{R}} (\rho\, i - \rho\, j)^2$ for all weak orders ρ on $\mathcal{X}$. Then for any β such that $0 < \beta < \frac{2E}{F} = \beta_0$ and for any ρ' not satisfying (6), $f(\rho^*) > f(\rho')$, and ρ' cannot be a solution. The theorem is proved.

Proof of Theorem 3: This proof is technical, and we give only a plan. Searching the minimum (7) subject to (8) and (9) with the Lagrange multiplier method, we get a system of n linear equations in $y_1, \ldots, y_n$. Summing all of them, we derive that the multiplier corresponding to (8) equals zero and then conclude that this system of equations for some ε coincides with that of the generalized row sum method. This completes the proof.

Proof of Theorem 4: This proof is very simple, however we give it in detail in order to illustrate the application of Self-Consistency. Assume on the contrary that there exists a strict aggregating operator that satisfies Self-Consistency.

1. Let $n > 2$. Consider the following array of paired comparisons $\mathcal{R}$: $r_{12}^1 = r_{23}^1 = r_{31}^1 = r_{\max}$; $r_{21}^1 = r_{32}^1 = r_{13}^1 = r_{\min}$; all other r_{ij}^p with $i \neq j$ are undefined (Figure A1).

Let ρ be a linear order on $\mathcal{X}$ in which $X_1 \succ_\rho X_2$ and $X_2 \succ_\rho X_3$. The set of comparison outcomes of X_1 is $\{(r_{12}^1 = r_{\max},\ r_{21}^1 = r_{\min}),\ (r_{13}^1 = r_{\min},\ r_{31}^1 = r_{\max})\}$; the set of comparison outcomes of X_3 is $\{(r_{31}^1 = r_{\max}, r_{13}^1 = r_{\min}), (r_{32}^1 = r_{\min}, r_{23}^1 = r_{\max})\}$. Note that (r_{31}^1, r_{13}^1) is stronger than (r_{12}^1, r_{21}^1) with respect to ρ, since $X_1 \succ_\rho X_2$, and (r_{32}^1, r_{23}^1) is stronger than (r_{13}^1, r_{31}^1) with respect to ρ, since $X_2 \succ_\rho X_3$.

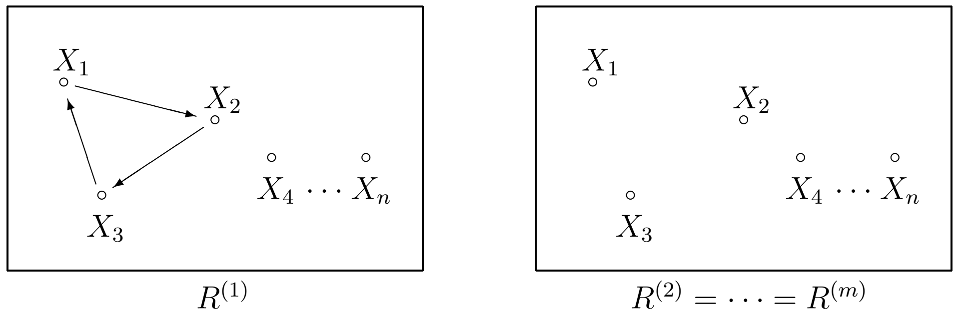


FIGURE A1. $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ in the proof of Theorem 4: $n > 2$. Every arc (X_i, X_j) designates that $r^p_{ij} = r_{\max}$ and $r^p_{ji} = r_{\min}$.

Therefore $X_1 \succ_\rho X_3$ contradicts Self-Consistency, and ρ is not optimal. Arguing as above we obtain that any linear order for which $X_2 \succ_\rho X_3 \succ_\rho X_1$ or $X_3 \succ_\rho X_1 \succ_\rho X_2$ is not optimal. Now assume $X_1 \succ_\rho X_3$ and $X_3 \succ_\rho X_2$. Let us compare the outcomes of X_2 and X_1. Note that (r^1_{23}, r^1_{32}) is stronger than (r^1_{12}, r^1_{12}) with respect to ρ, since $X_3 \succ_\rho X_2$ and (r^1_{21}, r^1_{12}) is stronger than (r^1_{13}, r^1_{31}), since $X_1 \succ_\rho X_3$. Therefore, $X_1 \succ_\rho X_2$ contradicts Self-Consistency, and ρ is not optimal. In the same way we conclude that if $X_2 \succ_\rho X_1 \succ_\rho X_3$ or $X_3 \succ_\rho X_2 \succ_\rho X_1$, then ρ is not optimal. Since every linear order on $\mathcal{X}$ obeys one of the above six assumptions, we obtain that the set of optimal orders is empty in contradiction to the definition of aggregating operator.

2. Let $m > 1$. Consider the following $\mathcal{R}$: $r^1_{12} = r^2_{21} = r_{\max}$; $r^1_{21} = r^2_{12} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A2).

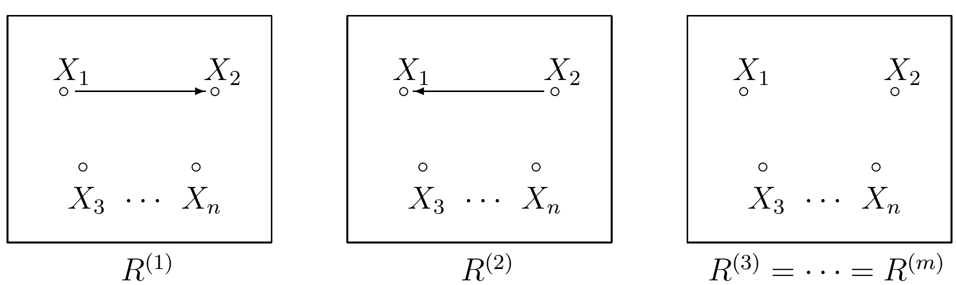


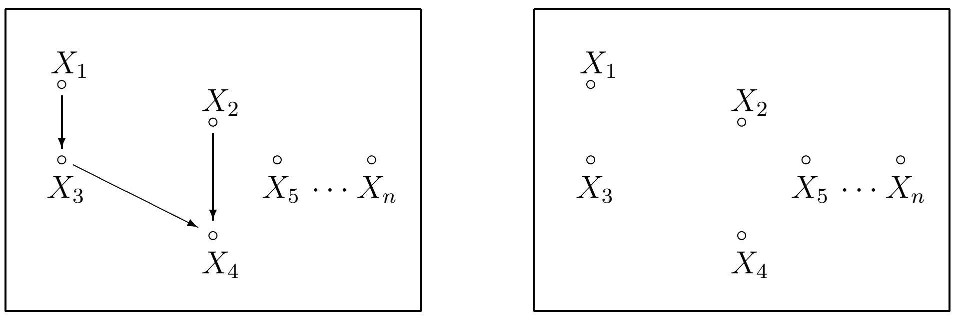
FIGURE A2. $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ in the proof of Theorem 4: $m > 1$. Every arc (X_i, X_j) designates that $r^p_{ij} = r_{\max}$ and $r^p_{ji} = r_{\min}$.

In the same way as above, we obtain that any optimal linear order can contain neither (X_1, X_2) nor (X_2, X_1), and the set of optimal orders is empty. Again we have contradiction with the definition of aggregating operator.

3. We have not covered the case $m = 1$, $n = 2$ yet. Here we can only offer the degenerate proof described in Section 8. If *draws* (i.e., $r^p_{ij} = r^p_{ji}$) are allowed, this provides a more sensible proof.

Proof of Theorem 5: Consider any aggregating operator that is indifferent to the degree of resulting preferences. Assume that it satisfies SCM.

1. Let $n > 3$. Consider the following $\mathcal{R} : r^1_{13} = r^1_{34} = r^1_{24} = r_{\max}$, $r^1_{31} = r^1_{43} = r^1_{42} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A3).



$R^{(1)}$ $\qquad$ $R^{(2)} = \cdots = R^{(m)}$

Figure A3. $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ in the proofs of Theorem 5 and Theorem 6: $n > 3$.

Let ρ be an optimal order for $\mathcal{R}$. Contrasting two alternatives in the manner described in the formulation of SCM will be called *confrontation.* Then

(A) Confronting X_1 and X_4 and using SCM, we get $X_1 \succ_\rho X_4$.

(B) Confronting X_3 and X_4 and assuming $X_4 \succeq_\rho X_3$, we get $X_3 \succ_\rho X_1$, in contradiction to (A). Therefore $X_3 \succ_\rho X_4$.

(C) Confronting X_2 and X_3, we get $X_2 \succ_\rho X_3$.

(D) Confronting X_1 and X_2 and using (B), we get $X_1 \succ_\rho X_2$.

Thus, it follows from SCM that the restriction of any optimal order to $\{X_1, X_2, X_3, X_4\}$ is the transitive closure of $\{(X_1, X_2), (X_2, X_3), (X_3, X_4)\}$.

Consider the weak order ρ' that is obtained from ρ by interchanging X_1 and X_2. Then for all r^p_{ij} defined in $\mathcal{R}$, $\mathrm{sign}(\rho\, i - \rho\, j) = \mathrm{sign}(\rho'\, i - \rho'\, j)$, and ρ' is optimal too, since the operator is indifferent to the degree of resulting preferences by our assumption. On the other hand, r' violates SCM (see (D)). This contradiction proves the desired statement.

2. $n = 3$. Consider the following $\mathcal{R}$: $r^1_{12} = r^1_{13} = r_{\max}$, $r^1_{21} = r^1_{31} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A4).

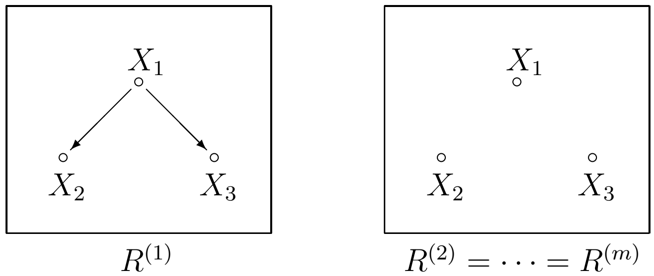


FIGURE A4. $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ in the proofs of Theorem 5 and Theorem 6: $n = 3$.

Let ρ be an optimal order for $\mathcal{R}$. Then using SCM and confronting X_1 and X_2, we have $X_1 \succ_\rho X_2$; confronting X_2 and X_3, we get $X_2 \sim_\rho X_3$. On the other hand, indifference to the degree of resulting preferences implies that the orders determined by $X_1 \succ_\rho X_2 \succ_\rho X_3$ and $X_1 \succ_\rho X_3 \succ_\rho X_2$ are optimal too, in contradiction to SCM. The theorem is proved. Finally, note that the latter argument can be extended to the case $n > 3$. However, we have preferred to give another proof for that case, since it demonstrates that indifference to the degree of resulting preferences allows (in some cases) to set $X_j \succ_\rho X_i$ whereas by SCM, $X_i \succ_\rho X_j$.

PROOF OF COROLLARY 1: It suffices to show that these operators are indifferent to the degree of resulting preferences. This is obvious.

PROOF OF THEOREM 6: Assume that one of these operators satisfies SCM.

1. Let $n > 3$. Consider the following $\mathcal{R}$, the same as in the proof of Theorem 5: $r^1_{13} = r^1_{34} = r^1_{24} = r_{\max}$, $r^1_{31} = r^1_{43} = r^1_{42} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A3). Let ρ be an optimal order for $\mathcal{R}$. As have been shown in the proof of Theorem 5, SCM implies that the restriction of any optimal order to $\{X_1, X_2, X_3, X_4\}$ is the transitive closure of $\{(X_1, X_2), (X_2, X_3), (X_3, X_4)\}$.

Consider the weak order ρ' that is obtained from ρ by interchanging X_1 and X_2. Let f_k be the objective functions of WQA(R, C'_k), $k \in \{4, 5, 6\}$ and let f_{NDW} be the objective function of "Net-Difference-WALB". Then for $\mathcal{R}$,

$$f_5(\rho) = (\rho\, 1 - \rho\, 3 + 1) r_{\max} + (\rho\, 2 - \rho\, 4 + 1) r_{\max} + (\rho\, 3 - \rho\, 4 + 1) r_{\max}$$

$$= (\rho\, 1 + \rho\, 2 - 2\, \rho\, 4 + 3) r_{\max} = f_5(\rho');$$

$$f_6(\rho) = (\rho\,3 - \rho\,1 - 1)r_{\min} + (\rho\,4 - \rho\,2 - 1)r_{\min} + (\rho\,4 - \rho\,3 - 1)r_{\min}$$
$$= (2\,\rho\,4 - \rho\,1 - \rho\,2 - 3)r_{\min} = f_6(\rho');$$

$f_4(\rho) = f_4(\rho')$ by Theorem 1, and

$$f_{\mathrm{NDW}}(\rho) = 0 = f_{\mathrm{NDW}}(\rho').$$

We see that all these operators equalize ρ' and ρ for $\mathcal{R}$, and thus ρ' is also optimal, which contradicts SCM.

2. $n = 3$. Consider the following $\mathcal{R}$, the same as in the proof of Theorem 5: $r^1_{12} = r^1_{13} = r_{\max}$, $r^1_{21} = r^1_{31} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A4). By SCM, a unique optimal weak order ρ is determined by $X_1 \succ_\rho X_2 \sim_\rho X_3$. On the other hand, each of the four operators under consideration equalizes ρ and the orders determined by $X_1 \succ_\rho X_2 \succ_\rho X_3$ and $X_1 \succ_\rho X_3 \succ_\rho X_2$. Hence they are also optimal, which contradicts SCM. The theorem is proved. Here the final remark in the proof of Theorem 5 is applicable as well.

Proof of Theorem 7: Let $n > 4$. Consider the following $\mathcal{R}$: for all $(i,j) \in \{(k,\ell) :\ k,\ell \in \{1,\dots,n\},\ k < \ell\} \setminus \{(1,2),(1,4),(2,3)\}$, $r^1_{ij} = r_{\max}$ and $r^1_{ji} = r_{\min}$; all other r^p_{ij} with $i \neq j$ are undefined (Figure A5).

Then $t_1 = t_2 = n-3$, $t_3 = n-4$, $t_4 = n-6$, and for $i = 5,\dots,n$, $t_i = n+1-2i$.

Let us prove that for $\mathcal{R}$ there exists only one weak order ρ satisfying SCM and that it is determined by $X_1 \succ_\rho X_2 \succ_\rho X_3 \succ_\rho X_4 \succ_\rho X_5 \succ_\rho \cdots \succ_\rho X_n$. Indeed, we have the following.

(A) For any $i \in \{1,\dots,4\}$ and $j \in \{5,\dots,n\}$, confronting X_i and X_j yields $X_i \succ_\rho X_j$.

(B) For any $i \in \{5,\dots,n-1\}$ and $j \in \{i+1,\dots,n\}$, confronting X_i and X_j yields $X_i \succ_\rho X_j$.

(C) Confronting X_1 and X_4, we get $X_1 \succ_\rho X_4$.

(D) Confronting X_2 and X_3, we get $X_2 \succ_\rho X_3$.

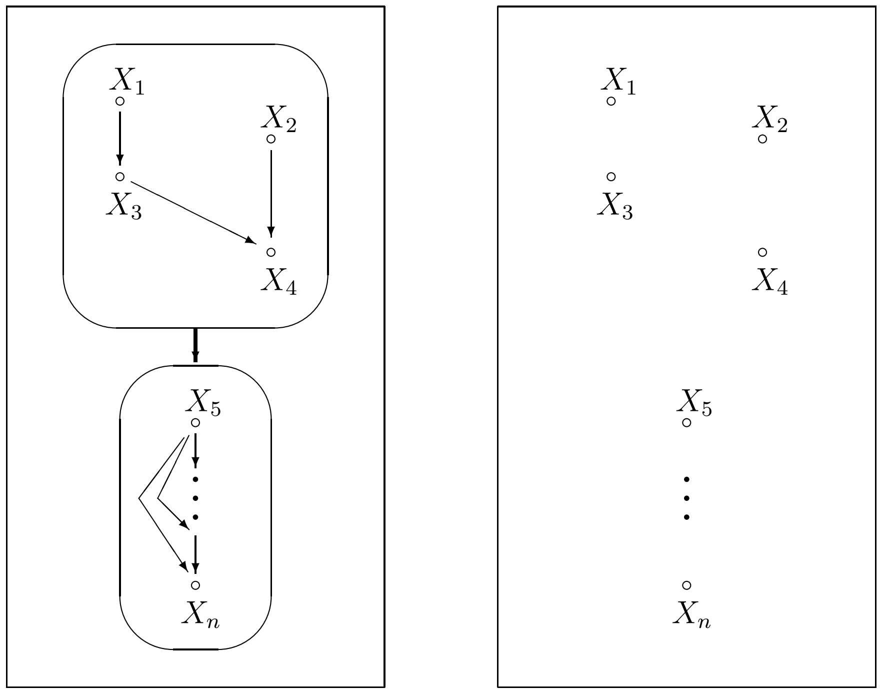
(E) Assuming $X_4 \succeq_\rho X_3$ and confronting X_3 and X_4, we get $X_3 \succ_\rho X_1$ in contradiction to (C). Therefore $X_3 \succ_\rho X_4$.

(F) Confronting X_1 and X_2 and using (E), we have $X_1 \succ_\rho X_2$.

Now consider ρ' determined by $X_1 \sim_{\rho'} X_2 \succ_{\rho'} X_3 \succ_{\rho'} X_4 \succ_{\rho'} \cdots \succ_{\rho'} X_n$ and let $f(\cdot)$ be the objective function of the β-LS method. It can be shown that for this $\mathcal{R}$

$$f(\rho) - f(\rho') = 4\beta^2(n-5).$$

Therefore, ρ is not uniquely optimal for $r = 5$ and is not optimal at all for $n > 5$. Hence the β-LS method violates Self-Consistency, and the theorem is proved.



$R^{(1)}$ $\qquad$ $R^{(2)} = \cdots = R^{(m)}$

FIGURE A5. $\mathcal{R} = (R^{(1)}, \ldots, R^{(m)})$ in the proof of Theorem 7. The bold arrow between two subsets designates that for every X_i in the first subset and every X_j in the second subset, $r_{ij}^p = r_{\max}$ and $r_{ji}^p = r_{\min}$.

PROOF OF THEOREM 8: Suppose that the conditions of the nonstrict part of Self-Consistent Monotonicity are satisfied but $X_j \succ_\rho X_i$, where ρ is the optimal weak order for the generalized row sum method. Consider the ith and jth equations of (4):

$$\begin{aligned} x_i &= \sum_{r_{ik}^p \in \mathcal{R}_i'} ((1+\varepsilon\, mn) r_{ik}^p + \varepsilon \cdot (x_k - x_i)) + \sum_{r_{ik}^p \in \mathcal{R}_i''} ((1+\varepsilon\, mn) r_{\max} + \varepsilon \cdot (x_k - x_i)), \\ x_j &= \sum_{r_{jk}^p \in \mathcal{R}_j'} ((1+\varepsilon\, mn) r_{jk}^p + \varepsilon \cdot (x_k - x_j)) + \sum_{r_{jk}^p \in \mathcal{R}_j''} ((1+\varepsilon\, mn) r_{\min} + \varepsilon \cdot (x_k - x_j)). \end{aligned}$$

For every $r_{ik}^p \in \mathcal{R}_i'$, by $r_{j\tilde{k}}^{\tilde{p}}$ denote the corresponding comparison outcome in $\mathcal{R}_j'$. After subtraction, we get

$$x_i - x_j = \sum_{r_{ik}^p \in \mathcal{R}_i'} ((1+\varepsilon\, mn)(r_{ik}^p - r_{j\tilde{k}}^{\tilde{p}}) + \varepsilon \cdot (x_k - x_{\tilde{k}}) + \varepsilon \cdot (x_j - x_i))$$

$$
\begin{aligned}
&+\sum_{r^p_{ik}\in\mathcal{R}''_i}((1+\varepsilon\, mn)r_{\max}+\varepsilon\cdot(x_k-x_i))\\
&+\sum_{r^p_{jk}\in\mathcal{R}''_j}((1+\varepsilon\, mn)r_{\max}+\varepsilon\cdot(x_j-x_k)).
\end{aligned}
$$

Suppose that ε is reasonable and positive. By our assumptions, all terms in the right-hand side are non-negative, whereas the left-hand side is negative. This contradiction proves that $X_i \succeq_\rho X_j$. The strict part of Self-Consistent Monotonicity and Self-Consistency can be proved in the same way.

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